Quantum Physics 1 - Test 1

Consider the wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t},$$

where A, λ, and ω are positive real constants.

a. Normalize $\Psi(x,t)$. *(2 points)*

b. What are the expectation values $\langle x \rangle$ and $\langle x^2 \rangle$? Explain your answer. *(3 points)*
Hint: You can use the integral $\int_0^\infty x^n e^{-x/a} dx = n! a^{n+1}$.

c. Find the standard deviation of x. *(1 points)*

d. To illustrate the sense in which σ_x represents the "spread" in x, sketch the graph of $|\Psi|^2$ as a function of x, and mark the points $(\langle x \rangle + \sigma_x)$ and $(\langle x \rangle - \sigma_x)$. *(3 points)*

e. *Bonus:* What is the probability that the particle would be found outside the domain from $(\langle x \rangle - \sigma_x)$ to $(\langle x \rangle + \sigma_x)$? *(1 point)*

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Quantum Physics 1 - Test 2

Consider a particle of mass m subject to the harmonic oscillator potential ($V(x) = \frac{1}{2}m\omega^2 x^2$), and assume that, at $t = 0$, the particle is in the state

$$\psi_0(x) = Ae^{-\frac{m\omega}{2\hbar}x^2}.$$

a. Normalize this wave function. (*1 point*)
   Hint: $\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}$.

b. Find the first excited state. You do not have to normalize it. (*2 points*)
   Hint:The ladder operator is given by $a_+ = \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)$.

c. Add time dependence to the initial state, i.e. find $\Psi_0(x,t)$. (*3 points*)

d. Find the allowed energies of the *half* harmonic oscillator (*3 points*)

$$V_{\frac{1}{2}}(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2, & \text{for } x > 0. \\ \infty, & \text{for } x < 0. \end{cases}$$

   (This represents, for example, a spring that can be streched, but not compressed.)
   Hint: This requires some careful thought, but very little actual computation.

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Quantum Physics 1 – Test 3

A free particle has the initial wave function

$$\Psi(x,0) = Ae^{-a|x|},$$

where A and a are positive real constants.

a. Normalize $\Psi(x,0)$. (*1 point*)

b. Sketch a graph of $|\Psi(x,0)|^2$. (*1 point*)

c. Find $\phi(k)$. (*4 points*)
Hint: $\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x,0)e^{-ikx}\mathrm{d}x$

d. Discuss the limiting cases where a is very small or very large. What can you say about the position and momentum in these cases? are they well- or ill-defined? (*3 points*)

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Quantum Physics 1 - Test 1

1. Consider the potential

$$V(x) = \begin{cases} 0, & \text{for } x < -a \\ V_0, & \text{for } -a \leq x \leq a \\ 0, & \text{for } x > a \end{cases}$$

Where V_0 is a positive real constant. See figure 1.

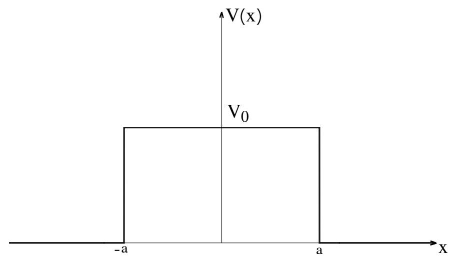

Figure 1: The rectangular barrier potential $V(x)$

a. Consider a particle on the left side of the barrier (so $x < -a$), traveling in the $+x$-direction. The energy of the particle is smaller than V_0. Is it possible to find the particle on the right side of the barrier according to classical physics? According to quantum physics? If there is a difference, what phenomenon causes this? *(1 point)*

b. Write down the three-piece wave function that matches the potential $V(x)$ for states with $E > V_0$. *(3 points)*

c. Using the (dis)continuity of the wave function at the boundaries, it can be derived that (if $E > V_0$):

$$T^{-1} = 1 + \frac{V_0^2}{4E(E - V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E - V_0)}\right)$$

For what energies will perfect transmission occur? *(1 point)*

2. Consider the operator

$$\hat{Q} \equiv i\frac{d}{d\phi},$$

where ϕ is the usual polar coordinate in two dimensions.
Hint: Work with functions $f(\phi)$ on the finite interval $0 \leq \phi \leq 2\pi$ and stipulate that $f(\phi + 2\pi) = f(\phi)$.

a. Is $\hat{Q}$ hermitian? *(2 points)*

b. Find its eigenfunctions and eigenvalues. *(2 points)*

Quantum Physics 1 - Test 5

a. Calculate $[p^2, x]$ (*1 points*)

b. Prove the virial theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle xp\rangle = 2\langle T\rangle - \left\langle x\frac{\mathrm{d}V}{\mathrm{d}x}\right\rangle$$

where $T = \frac{\hat{p}^2}{2m}$ is the kinetic energy ($H = T + V$). (*3 points*)
Hint: use that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Q\rangle = \frac{i}{\hbar}\left\langle\left[\hat{H}, \hat{Q}\right]\right\rangle + \left\langle\frac{\partial \hat{Q}}{\partial t}\right\rangle$$

c. What does this reduce to for stationary states? (*1 point*)

d. For the harmonic oscillator ($V = \frac{1}{2}m\omega^2x^2$), find the relation between $\langle T\rangle$ and $\langle V\rangle$ (*1 point*)

e. Are the following statements true or false? (*3 points*)

1. A state cannot be a simultaneous eigenstate of two compatible observables.
2. Hermitian operators have real eigenvalues in finite-dimensional vector spaces.
3. All wavefunctions are linear combinations of eigenfunctions of a Hermitian operator.
4. The generalized uncertainty principle is one of the central assumptions of quantum mechanics.
5. If a matrix is equal to the complex conjugate of its transpose, then the matrix is Hermitian.
6. Not all observables are Hermitian.

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Quantum Physics 1 - Test 6

Consider the three-dimensional harmonic oscillator, for which the potential is

$$V(r) = \frac{1}{2} m\omega^2 r^2. \tag{1}$$

a. Show that the separation of variables in cartesian coordinates turns this into three one-dimensional oscillators. (*2 points*)

   Hint: Time-independent Schrödinger equation for the one-dimensional harmonic oscillator

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{d^2x} + V\psi = E\psi \tag{2}$$

b. Derive the allowed energies of the total wavefunction of the three-dimensional harmonic oscillator using what found in the previous point. (*2 points*)

c. Given the ground state of the three-dimensional harmonic oscillator

$$\psi_0(r) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-\frac{m\omega}{2\hbar}r^2}, \tag{3}$$

   compute $\langle r \rangle$. (*2 points*)

   Hint:

$$\int_0^\infty \mathrm{d}x\, x^{2n+1} e^{-x^2/a^2} = \frac{n!}{2} a^{2n+2}. \tag{4}$$

d. Are the following statements true or false? (*3 points*)

   1. The ground state of the system above has non-zero $\langle \mathbf{p} \rangle$.
   2. The first excited state of the system above is 3-fold degenerate.
   3. The Coulomb potential that describes proton-electron interaction admits both continuum and bound states.

Quantum Physics 1 - Test 7

An electron is in the spin state

$$\chi = A \begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix}.$$

a. Determine the constant A by normalizing χ. (*1 point*)

b. If you measured S_z on this electron, what values could you get, and what is the probability of each value? What is the expectation value of S_z? (*3 points*)

c. Show that the normalized eigenspinors of S_x are given by $\chi_+^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\chi_-^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. (*2 points*)
Hint: $S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

d. Suppose we reset the system to its original state, if you measured S_x on the electron, what values could you get, and what is the probability of each value? What is the expectation value of S_x? (*3 points*)

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Quantum Physics 1 - Test 8

1. Consider two electrons (spin 1/2).

a. Give the two-electron spin states ($|s\,m\rangle$) in terms of the one-electron spin states. i.e. give the triplet and singlet states. *(2 points)*

b. What are entangled state in general *and* which of the states of a. are entangled? *(2 points)*

c. Consider the two hydrogen atoms which form a covalent bond by sharing their electrons. Do all spin states found in a. allow for this configuration? Motivate you answer *(2 points)*

2. For the following statements write down "true" when the statement is correct and "false" when the statement is incorrect. Motivate your answer with a single sentence. *(3 points)*

a. Two electrons cannot occupy the same state.

b. Nickel ($^3F_4$) has the quantum numbers: $S = 3$, $L = 3$, $J = 4$

c. Quantum physics contains local hidden variables

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